

विध्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचित: मानव धर्म प्रणेता
सद्गुरु श्री रणछोड़दासजी महाराज

STUDY PACKAGE

Subject : Mathematics

Topic : Properties of Triangle



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1. Theory
2. Short Revision
3. Exercise (Ex. 3 + 2 = 5)
4. Assertion & Reason
5. Que. from Compt. Exams
6. 39 Yrs. Que. from IIT-JEE
7. 15 Yrs. Que. from AIEEE

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Properties & Solution of Triangle

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1. Sine Rule:

In any triangle ABC, the sines of the angles are proportional to the opposite sides i.e.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example : In any ΔABC , prove that $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

Solution. \therefore We have to prove $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

\therefore From **sine rule**, we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (let)}$$

$\Rightarrow a = k \sin A, b = k \sin B$ and $c = k \sin C$

$$\therefore \text{L.H.S.} = \frac{a+b}{c}$$

$$= \frac{k(\sin A + \sin B)}{k \sin C}$$

$$= \frac{\cos\frac{C}{2} \cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2} \cos\frac{C}{2}}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

= R.H.S. Hence L.H.S. = R.H.S. **Proved**

$$= \frac{\sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2} \cos\frac{C}{2}}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

Example : In any ΔABC , prove that $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Solution. \therefore We have to prove that $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$
 \therefore from **sine rule**, we know that $a = k \sin A, b = k \sin B$ and $c = k \sin C$
 $\therefore (b^2 - c^2) \cot A = k^2 (\sin^2 B - \sin^2 C) \cot A$
 $\therefore \sin^2 B - \sin^2 C = \sin(B+C) \sin(B-C)$
 $\therefore (b^2 - c^2) \cot A = k^2 \sin(B+C) \sin(B-C) \cot A$

$$\therefore B + C = \pi - A$$

$$\therefore (b^2 - c^2) \cot A = k^2 \sin A \sin(B-C) \frac{\cos A}{\sin A}$$

$$= -k^2 \sin(B-C) \cos(B+C)$$

$$= -\frac{k^2}{2} [2 \sin(B-C) \cos(B+C)]$$

$$\therefore \cos A = -\cos(B+C)$$

$$\Rightarrow (b^2 - c^2) \cot A = -\frac{k^2}{2} [\sin 2B - \sin 2C] \dots\dots\dots(i)$$

$$\text{Similarly } (c^2 - a^2) \cot B = -\frac{k^2}{2} [\sin 2C - \sin 2A] \dots\dots\dots(ii)$$

$$\text{and } (a^2 - b^2) \cot C = -\frac{k^2}{2} [\sin 2A - \sin 2B] \dots\dots\dots(iii)$$

adding equations (i), (ii) and (iii), we get $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$ **Hence Proved**

Self Practice Problems

In any ΔABC , prove that

1. $a \sin\left(\frac{A}{2} + B\right) = (b + c) \sin\left(\frac{A}{2}\right)$.

2. $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$

3. $\frac{c}{a-b} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$.

2. Cosine Formula:

$$(i) \cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{or} \quad a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 + 2bc \cos (B + C)$$

$$(ii) \cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad (iii) \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example : In a triangle ABC if a = 13, b = 8 and c = 7, then find sin A.

Solution. $\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 49 - 169}{2 \cdot 8 \cdot 7}$

$$\Rightarrow \cos A = -\frac{1}{2} \qquad \Rightarrow \qquad A = \frac{2\pi}{3}$$

$$\therefore \sin A = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \text{Ans.}$$

***Example :** In a ΔABC , prove that $a(b \cos C - c \cos B) = b^2 - c^2$

Solution. \therefore We have to prove $a(b \cos C - c \cos B) = b^2 - c^2$.
 \therefore from **cosine rule** we know that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \& \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\begin{aligned} \therefore \text{L.H.S.} &= a \left\{ b \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - c \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right\} \\ &= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} \\ &= \frac{b^2 - c^2}{2} = \text{R.H.S.} \end{aligned}$$

Hence L.H.S. = R.H.S. **Proved**

Example : If in a ΔABC , $\angle A = 60^\circ$ then find the value of $\left(1 + \frac{a}{c} + \frac{b}{c}\right) \left(1 + \frac{c}{b} - \frac{a}{b}\right)$.

Solution. $\therefore \angle A = 60^\circ$

$$\begin{aligned} \therefore \left(1 + \frac{a}{c} + \frac{b}{c}\right) \left(1 + \frac{c}{b} - \frac{a}{b}\right) &= \left(\frac{c+a+b}{c}\right) \left(\frac{b+c-a}{b}\right) \\ &= \frac{(b+c)^2 - a^2}{bc} \\ &= \frac{(b^2 + c^2 - a^2) + 2bc}{bc} \\ &= \frac{b^2 + c^2 - a^2}{bc} + 2 \\ &= 2 \left(\frac{b^2 + c^2 - a^2}{2bc}\right) + 2 \\ &= 2\cos A + 2 \qquad \because \angle A = 60^\circ \Rightarrow \cos A = \frac{1}{2} \end{aligned}$$

$$\therefore \left(1 + \frac{a}{c} + \frac{b}{c}\right) \left(1 + \frac{c}{b} - \frac{a}{b}\right) = 3 \text{ Ans.}$$

Self Practice Problems :

- The sides of a triangle ABC are a, b, $\sqrt{a^2 + ab + b^2}$, then prove that the greatest angle is 120° .
- In a triangle ABC prove that $a(\cos B + \cos C) = 2(b + c) \sin^2 \frac{A}{2}$.

3. Projection Formula:

$$(i) a = b \cos C + c \cos B \qquad (ii) b = c \cos A + a \cos C \qquad (iii) c = a \cos B + b \cos A$$

Example : In a triangle ABC prove that $a(b \cos C - c \cos B) = b^2 - c^2$

Solution. \therefore L.H.S. = $a(b \cos C - c \cos B)$
 $= b(a \cos C) - c(a \cos B)$ (i)

\therefore From **projection rule**, we know that

$$\begin{aligned} b &= a \cos C + c \cos A \qquad \Rightarrow \qquad a \cos C = b - c \cos A \\ \& \quad c &= a \cos B + b \cos A \qquad \Rightarrow \qquad a \cos B = c - b \cos A \end{aligned}$$

Put values of a cos C and a cos B in equation (i), we get

$$\begin{aligned} \text{L.H.S.} &= b(b - c \cos A) - c(c - b \cos A) \\ &= b^2 - bc \cos A - c^2 + bc \cos A \\ &= b^2 - c^2 \\ &= \text{R.H.S.} \end{aligned}$$

Hence L.H.S. = R.H.S. **Proved**

Note: We have also proved $a(b \cos C - c \cos B) = b^2 - c^2$ by using **cosine - rule** in solved ***Example**.

Example : In a ΔABC prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

Solution. \therefore L.H.S. = $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C$

$$\begin{aligned}
 &= b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C \\
 &= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (c \cos B + b \cos C) \\
 &= a + b + c \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence L.H.S. = R.H.S. **Proved**

Self Practice Problems

In a ΔABC , prove that

1. $2 \left(b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} \right) = a + b + c.$

2. $\frac{\cos B}{\cos C} = \frac{c - b \cos A}{b - c \cos A}.$

3. $\frac{\cos A}{c \cos B + b \cos C} + \frac{\cos B}{a \cos C + c \cos A} + \frac{\cos C}{a \cos B + b \cos A} = \frac{a^2 + b^2 + c^2}{2abc}.$

4. Napier's Analogy - tangent rule:

(i) $\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$

(ii) $\tan \frac{C - A}{2} = \frac{c - a}{c + a} \cot \frac{B}{2}$

(iii) $\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$

Example : Find the unknown elements of the ΔABC in which $a = \sqrt{3} + 1, b = \sqrt{3} - 1, C = 60^\circ$.

Solution. $\therefore a = \sqrt{3} + 1, b = \sqrt{3} - 1, C = 60^\circ$

$\therefore A + B + C = 180^\circ$

$\therefore A + B = 120^\circ$ (i)

\therefore From law of tangent, we know that

$$\begin{aligned}
 \tan \left(\frac{A - B}{2} \right) &= \frac{a - b}{a + b} \cot \frac{C}{2} \\
 &= \frac{(\sqrt{3} + 1) - (\sqrt{3} - 1)}{(\sqrt{3} + 1) + (\sqrt{3} - 1)} \cot 30^\circ \\
 &= \frac{2}{2\sqrt{3}} \cot 30^\circ
 \end{aligned}$$

$\Rightarrow \tan \left(\frac{A - B}{2} \right) = 1$

$\therefore \frac{A - B}{2} = \frac{\pi}{4} = 45^\circ$

$\Rightarrow A - B = 90^\circ$ (ii)

From equation (i) and (ii), we get $A = 105^\circ$ and $B = 15^\circ$

Now,

\therefore From **sine-rule**, we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\begin{aligned}
 \therefore c &= \frac{a \sin C}{\sin A} = \frac{(\sqrt{3} + 1) \sin 60^\circ}{\sin 105^\circ} \\
 &= \frac{(\sqrt{3} + 1) \frac{\sqrt{3}}{2}}{\frac{\sqrt{3} + 1}{2\sqrt{2}}} \quad \therefore \sin 105^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}
 \end{aligned}$$

$\Rightarrow c = \sqrt{6}$

$\therefore c = \sqrt{6}, A = 105^\circ, B = 15^\circ$ **Ans.**

Self Practice Problem

1. In a ΔABC if $b = 3, c = 5$ and $\cos(B - C) = \frac{7}{25}$, then find the value of $\tan \frac{A}{2}$.

Ans. $\frac{1}{3}$

2. If in a ΔABC , we define $x = \tan\left(\frac{B-C}{2}\right) \tan\frac{A}{2}$, $y = \tan\left(\frac{C-A}{2}\right) \tan\frac{B}{2}$ and $z = \tan\left(\frac{A-B}{2}\right) \tan\frac{C}{2}$ then show that $x + y + z = -xyz$.

5. Trigonometric Functions of Half Angles:

- (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$
- (ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$
- (iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ is semi perimetre of triangle.
- (iv) $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

6. Area of Triangle (Δ)

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

Example : In a ΔABC if a, b, c are in A.P. then find the value of $\tan\frac{A}{2} \cdot \tan\frac{C}{2}$.

Solution. $\therefore \tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$ and $\tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

$\therefore \tan\frac{A}{2} \cdot \tan\frac{C}{2} = \frac{\Delta^2}{s^2(s-a)(s-c)}$ $\therefore \Delta^2 = s(s-a)(s-b)(s-c)$

$\therefore \tan\frac{A}{2} \cdot \tan\frac{C}{2} = \frac{s-b}{s} = 1 - \frac{b}{s}$ (i)

\therefore it is given that a, b, c are in A.P.

$\Rightarrow 2b = a + c$

$\therefore s = \frac{a+b+c}{2} = \frac{3b}{2}$

$\therefore \frac{b}{s} = \frac{2}{3}$ put in equation (i)

$\therefore \tan\frac{A}{2} \cdot \tan\frac{C}{2} = 1 - \frac{2}{3}$

$\Rightarrow \tan\frac{A}{2} \cdot \tan\frac{C}{2} = \frac{1}{3}$ **Ans.**

Example : In a ΔABC if $b \sin C (b \cos C + c \cos B) = 42$, then find the area of the ΔABC .

Solution. $\therefore b \sin C (b \cos C + c \cos B) = 42$ (i) given

\therefore From **projection rule**, we know that

$a = b \cos C + c \cos B$ put in (i), we get

$ab \sin C = 42$ (ii)

$\therefore \Delta = \frac{1}{2} ab \sin C$

$\therefore \Delta = 21$ sq. unit **Ans.**

Example : In any ΔABC prove that $(a + b + c) \left(\tan\frac{A}{2} + \tan\frac{B}{2} \right) = 2c \cot\frac{C}{2}$.

Solution. \therefore L.H.S. = $(a + b + c) \left(\tan\frac{A}{2} + \tan\frac{B}{2} \right)$

$\therefore \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

\therefore L.H.S. = $(a + b + c) \left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \right]$

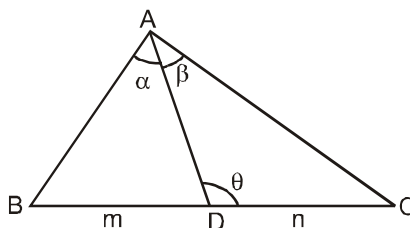
$= 2s \sqrt{\frac{s-c}{s}} \left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}} \right]$

$$\begin{aligned}
 &= 2 \sqrt{s(s-c)} \left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}} \right] & \because 2s = a + b + c \\
 & & \because 2s - b - a = c \\
 &= 2 \sqrt{s(s-c)} \left[\frac{c}{\sqrt{(s-a)(s-b)}} \right] \\
 &= 2c \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} & \because \cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \\
 &= 2c \cot \frac{C}{2} \\
 &= \text{R.H.S.} \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence L.H.S. = R.H.S.

Proved

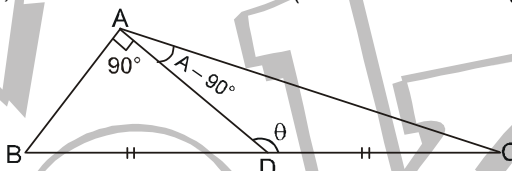
7. m - n Rule:



$$\begin{aligned}
 (m+n) \cot \theta &= m \cot \alpha - n \cot \beta \\
 &= n \cot B - m \cot C
 \end{aligned}$$

Example : If the median AD of a triangle ABC is perpendicular to AB, prove that $\tan A + 2 \tan B = 0$.

Solution. From the figure, we see that $\theta = 90^\circ + B$ (as θ is external angle of $\triangle ABD$)



Now if we apply **m-n rule** in $\triangle ABC$, we get
 $(1 + 1) \cot (90 + B) = 1 \cdot \cot 90^\circ - 1 \cdot \cot (A - 90^\circ)$
 $\Rightarrow -2 \tan B = \cot (90^\circ - A)$
 $\Rightarrow -2 \tan B = \tan A$
 $\Rightarrow \tan A + 2 \tan B = 0$ **Hence proved.**

Example : The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that

$$4 \left(1 + \frac{1}{t_2^2} \right) = \left(\frac{1}{t_1} + \frac{1}{t_2} \right) \left(\frac{1}{t_2} + \frac{1}{t_3} \right).$$

Solution. Let point D and E divides the base BC into three equal parts i.e. $BD = DE = DC = d$ (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex.

$\Rightarrow t_1 = \tan \alpha, t_2 = \tan \beta$ and $t_3 = \tan \gamma$
 Now in $\triangle ABC$

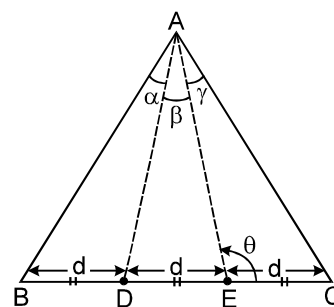
$$\begin{aligned}
 \because BE : EC &= 2d : d = 2 : 1 \\
 \therefore \text{from m-n rule, we get} \\
 (2 + 1) \cot \theta &= 2 \cot (\alpha + \beta) - \cot \gamma \\
 \Rightarrow 3 \cot \theta &= 2 \cot (\alpha + \beta) - \cot \gamma \quad \dots\dots\dots(i)
 \end{aligned}$$

again

$$\begin{aligned}
 \therefore \text{in } \triangle ADC \\
 \because DE : EC &= x : x = 1 : 1 \\
 \therefore \text{if we apply m-n rule in } \triangle ADC, \text{ we get} \\
 (1 + 1) \cot \theta &= 1 \cdot \cot \beta - 1 \cdot \cot \gamma \\
 2 \cot \theta &= \cot \beta - \cot \gamma \quad \dots\dots\dots(ii)
 \end{aligned}$$

from (i) and (ii), we get

$$\begin{aligned}
 \frac{3 \cot \theta}{2 \cot \theta} &= \frac{2 \cot (\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma} \\
 \Rightarrow 3 \cot \beta - 3 \cot \gamma &= 4 \cot (\alpha + \beta) - 2 \cot \gamma \\
 \Rightarrow 3 \cot \beta - \cot \gamma &= 4 \cot (\alpha + \beta) \\
 \Rightarrow 3 \cot \beta - \cot \gamma &= 4 \left\{ \frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha} \right\} \\
 \Rightarrow 3 \cot^2 \beta + 3 \cot \alpha \cot \beta - \cot \beta \cot \gamma - \cot \alpha \cot \gamma &= 4 \cot \alpha \cot \beta - 4 \\
 \Rightarrow 4 + 3 \cot^2 \beta &= \cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \alpha \cot \gamma \\
 \Rightarrow 4 + 4 \cot^2 \beta &= \cot \alpha \cot \beta + \cot \alpha \cot \gamma + \cot \beta \cot \gamma + \cot^2 \beta \\
 \Rightarrow 4(1 + \cot^2 \beta) &= (\cot \alpha + \cot \beta) (\cot \beta + \cot \gamma) \\
 \Rightarrow 4 \left(1 + \frac{1}{\tan^2 \beta} \right) &= \left(\frac{1}{\tan \alpha} + \frac{1}{\tan \beta} \right) \left(\frac{1}{\tan \beta} + \frac{1}{\tan \gamma} \right)
 \end{aligned}$$



$$\Rightarrow 4 \left(1 + \frac{1}{t_2^2}\right) = \left(\frac{1}{t_1} + \frac{1}{t_2}\right) \left(\frac{1}{t_2} + \frac{1}{t_3}\right) \quad \text{Hence proved}$$

Self Practice Problems :

1. In a ΔABC , the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides angle A into the angles of 30° and 45° . Prove that the side BC is of length 2 units.

8. Radius of Circumcircle :

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

Example : In a ΔABC prove that $\sin A + \sin B + \sin C = \frac{s}{R}$

Solution. In a ΔABC , we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\therefore \sin A = \frac{a}{2R}, \sin B = \frac{b}{2R} \text{ and } \sin C = \frac{c}{2R}.$$

$$\begin{aligned} \therefore \sin A + \sin B + \sin C &= \frac{a+b+c}{2R} & \because a+b+c &= 2s \\ &= \frac{2s}{2R} & \Rightarrow \sin A + \sin B + \sin C &= \frac{s}{R}. \end{aligned}$$

Example : In a ΔABC if $a = 13$ cm, $b = 14$ cm and $c = 15$ cm, then find its circumradius.

Solution. $\therefore R = \frac{abc}{4\Delta} \dots\dots(i)$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore s = \frac{a+b+c}{2} = 21 \text{ cm}$$

$$\therefore \Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7^2 \cdot 4^2 \cdot 3^2}$$

$$\Rightarrow \Delta = 84 \text{ cm}^2$$

$$\therefore R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8} \text{ cm}$$

$$\therefore R = \frac{65}{8} \text{ cm.}$$

Example : In a ΔABC prove that $s = 4R \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$.

Solution. In a ΔABC ,

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} \text{ and } \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} \text{ and } R = \frac{abc}{4\Delta}$$

$$\therefore \text{R.H.S.} = 4R \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}.$$

$$\begin{aligned} &= \frac{abc}{\Delta} \cdot s \sqrt{\frac{s(s-a)(s-b)(s-c)}{(abc)^2}} & \because \Delta &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \frac{s}{\Delta} \cdot s & & \\ &= \text{L.H.S.} & & \end{aligned}$$

Hence R.H.L = L.H.S. proved

Example : In a ΔABC , prove that $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$.

Solution. $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$

$$\therefore \text{L.H.S.} = \left(\frac{1}{s-a} + \frac{1}{s-b}\right) + \left(\frac{1}{s-c} - \frac{1}{s}\right)$$

$$= \frac{2s-a-b}{(s-a)(s-b)} + \frac{(s-s+c)}{s(s-c)} \quad \because 2s = a + b + c$$

$$= \frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}$$

$$= c \left[\frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right] = c \left[\frac{2s^2 - s(a+b+c) + ab}{\Delta^2} \right]$$

$$\therefore \text{L.H.S.} = c \left[\frac{2s^2 - s(2s) + ab}{\Delta^2} \right] = \frac{abc}{\Delta^2} = \frac{4R\Delta}{\Delta^2} = \frac{4R}{\Delta} \quad \therefore R = \frac{abc}{4\Delta}$$

$$\Rightarrow abc = 4R\Delta$$

$$\therefore \text{L.H.S.} = \frac{4R}{\Delta}$$

Self Practice Problems :

- In a ΔABC , prove the followings :
- $a \cot A + b \cot B + c \cot C = 2(R + r)$.
 - $4 \left(\frac{s}{a} - 1 \right) \left(\frac{s}{b} - 1 \right) \left(\frac{s}{c} - 1 \right) = \frac{r}{R}$.
 - If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha + \beta + \gamma} = r^2$

9. Radius of The Incircle :

$$(i) r = \frac{\Delta}{s} \qquad (ii) r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$$

$$(iii) r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \text{ \& so on} \qquad (iv) r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

10. Radius of The Ex-Circles :

$$(i) r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c} \qquad (ii) r_1 = s \tan \frac{A}{2}; r_2 = s \tan \frac{B}{2}; r_3 = s \tan \frac{C}{2}$$

$$(iii) r_1 = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} \text{ \& so on} \qquad (iv) r_1 = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$$

Example : In a ΔABC , prove that $r_1 + r_2 + r_3 - r = 4R = 2a \operatorname{cosec} A$

Solution. $\therefore \text{L.H.S} = r_1 + r_2 + r_3 - r$

$$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s}$$

$$= \Delta \left(\frac{1}{s-a} + \frac{1}{s-b} \right) + \Delta \left(\frac{1}{s-c} - \frac{1}{s} \right)$$

$$= \Delta \left[\frac{(s-b+s-a)}{(s-a)(s-b)} + \frac{(s-s+c)}{s(s-c)} \right]$$

$$= \Delta \left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right]$$

$$= c\Delta \left[\frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right]$$

$$= c\Delta \left[\frac{2s^2 - s(a+b+c) + ab}{\Delta^2} \right] \qquad \therefore a + b + c = 2s$$

$$= \frac{abc}{\Delta} \qquad \therefore R = \frac{abc}{4\Delta}$$

$$= 4R = 2a \operatorname{cosec} A \qquad \therefore \frac{a}{\sin A} = 2R = a \operatorname{cosec} A$$

= R.H.S.
Hence L.H.S. = R.H.S. **proved**

Example : If the area of a ΔABC is 96 sq. unit and the radius of the escribed circles are respectively 8, 12 and 24. Find the perimeter of ΔABC .

Solution. $\therefore \Delta = 96$ sq. unit
 $r_1 = 8, r_2 = 12$ and $r_3 = 24$

$$\therefore r_1 = \frac{\Delta}{s-a} \Rightarrow s - a = 12 \qquad \dots\dots(i)$$

$$\therefore r_2 = \frac{\Delta}{s-b} \Rightarrow s - b = 8 \qquad \dots\dots(ii)$$

$$\therefore r_3 = \frac{\Delta}{s-c} \Rightarrow s - c = 4 \qquad \dots\dots(iii)$$

\therefore adding equations (i), (ii) & (iii), we get

$$3s - (a + b + c) = 24$$

$$s = 24$$

$$\therefore \text{perimeter of } \triangle ABC = 2s = 48 \text{ unit.}$$

Self Practice Problems

In a $\triangle ABC$ prove that

- $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$
- $rr_1 + rr_2 + rr_3 = ab + bc + ca - s^2$
- If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a $\triangle ABC$, then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.
- $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

11. Length of Angle Bisectors, Medians & Altitudes :

(i) Length of an angle bisector from the angle $A = \beta_a = \frac{2bc \cos \frac{A}{2}}{b + c}$;

(ii) Length of median from the angle $A = m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$

& (iii) Length of altitude from the angle $A = A_a = \frac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} (a^2 + b^2 + c^2)$

Example : AD is a median of the $\triangle ABC$. If AE and AF are medians of the triangles ABD and ADC respectively, and $AD = m_1, AE = m_2, AF = m_3$, then prove that $m_2^2 + m_3^2 - 2m_1^2 = \frac{a^2}{8}$.

Solution.

\therefore In $\triangle ABC$

$$AD^2 = \frac{1}{4} (2b^2 + 2c^2 - a^2) = m_1^2 \quad \dots\dots(i)$$

$$\therefore \text{In } \triangle ABD, AE^2 = m_2^2 = \frac{1}{4} (2c^2 + 2AD^2 - \frac{a^2}{4}) \quad \dots\dots(ii)$$

$$\text{Similarly in } \triangle ADC, AF^2 = m_3^2 = \frac{1}{4} (2AD^2 + 2b^2 - \frac{a^2}{4}) \quad \dots\dots(iii)$$

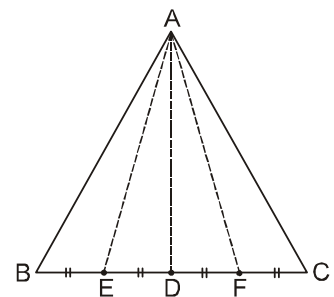
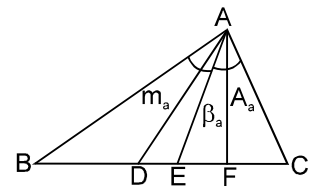
by adding equations (ii) and (iii), we get

$$\begin{aligned} \therefore m_2^2 + m_3^2 &= \frac{1}{4} \left(4AD^2 + 2b^2 + 2c^2 - \frac{a^2}{2} \right) \\ &= AD^2 + \frac{1}{4} \left(2b^2 + 2c^2 - \frac{a^2}{2} \right) \\ &= AD^2 + \frac{1}{4} \left(2b^2 + 2c^2 - a^2 + \frac{a^2}{2} \right) \\ &= AD^2 + \frac{1}{4} (2b^2 + 2c^2 - a^2) + \frac{a^2}{8} \\ &= AD^2 + AD^2 + \frac{a^2}{8} \\ &= 2AD^2 + \frac{a^2}{8} \end{aligned}$$

$$\therefore AD^2 = m_1^2$$

$$\therefore m_2^2 + m_3^2 - 2m_1^2 = \frac{a^2}{8}$$

Hence Proved



Self Practice Problem :

3. In a ΔABC $a = 5, b = 4, c = 3$. 'G' is the centroid of triangle, then find circumradius of ΔGAB .

Ans. $\frac{5}{12} \sqrt{13}$

12. The Distances of The Special Points from Vertices and Sides of Triangle:

- (i) Circumcentre (O) : $OA = R$ & $O_a = R \cos A$
- (ii) Incentre (I) : $IA = r \operatorname{cosec} \frac{A}{2}$ & $I_a = r$
- (iii) Excentre (I_1) : $I_1 A = r_1 \operatorname{cosec} \frac{A}{2}$ & $I_{1a} = r_1$
- (iv) Orthocentre (H) : $HA = 2R \cos A$ & $H_a = 2R \cos B \cos C$
- (v) Centroid (G) : $GA = \frac{1}{3} \sqrt{2b^2 + 2c^2 - a^2}$ & $G_a = \frac{2\Delta}{3a}$

Example : If x, y and z are respectively the distances of the vertices of the ΔABC from its orthocentre, then prove that

(i) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$ (ii) $x + y + z = 2(R + r)$

Solution.

$\therefore x = 2R \cos A, y = 2R \cos B, z = 2R \cos C$ and
and $a = 2R \sin A, b = 2R \sin B, c = 2R \sin C$

$\therefore \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \tan A + \tan B + \tan C$ (i)

& $\frac{abc}{xyz} = \tan A \cdot \tan B \cdot \tan C$ (ii)

\therefore We know that in a ΔABC $\Sigma \tan A = \Pi \tan A$
 \therefore From equations (i) and (ii), we get

$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$

$\therefore x + y + z = 2R (\cos A + \cos B + \cos C)$

\therefore in a ΔABC $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\therefore x + y + z = 2R \left(1 + 4 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \right)$

$= 2 \left(R + 4R \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \right) \therefore r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\therefore x + y + z = 2(R + r)$

Self Practice Problems

1. If I be the incentre of ΔABC , then prove that $IA \cdot IB \cdot IC = abc \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.
2. If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC , then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.

13. Orthocentre and Pedal Triangle:

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A, \pi - 2B$ and $\pi - 2C$.

(ii) Its sides are $a \cos A = R \sin 2A,$
 $b \cos B = R \sin 2B$ and
 $c \cos C = R \sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

14. Excentral Triangle:

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1 I_2 I_3$. (ii) Its angles are

$\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ & $\frac{\pi}{2} - \frac{C}{2}$.

- (iii) Its sides are $4R \cos \frac{A}{2}$,
 $4R \cos \frac{B}{2}$ & $4R \cos \frac{C}{2}$.
- (iv) $I I_1 = 4R \sin \frac{A}{2}$;
 $I I_2 = 4R \sin \frac{B}{2}$; $I I_3 = 4R \sin \frac{C}{2}$.
- (v) Incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1 I_2 I_3$.

15. Distance Between Special Points :

- (i) Distance between circumcentre and orthocentre
 $OH^2 = R^2 (1 - 8 \cos A \cos B \cos C)$
- (ii) Distance between circumcentre and incentre

$$OI^2 = R^2 (1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}) = R^2 - 2Rr$$

- (iii) Distance between circumcentre and centroid

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

Example : In I is the incentre and I_1, I_2, I_3 are the centres of escribed circles of the ΔABC , prove that

- (i) $II_1 \cdot II_2 \cdot II_3 = 16R^2 r$ (ii) $II_1^2 + I_2 I_3^2 = II_2^2 + I_3 I_1^2 = II_3^2 + I_1 I_2^2$

Solution. (i) \therefore We know that

$$II_1 = a \sec \frac{A}{2}, II_2 = b \sec \frac{B}{2} \text{ and } II_3 = c \sec \frac{C}{2}$$

$$\therefore I_1 I_2 = c \operatorname{cosec} \frac{C}{2}, I_2 I_3 = a \operatorname{cosec} \frac{A}{2} \text{ and } I_3 I_1 = b \operatorname{cosec} \frac{B}{2}$$

$$\therefore II_1 \cdot II_2 \cdot II_3 = abc \sec \frac{A}{2} \sec \frac{B}{2} \sec \frac{C}{2} \dots\dots(i)$$

$$\therefore a = 2R \sin A, b = 2R \sin B \text{ and } c = 2R \sin C$$

$$\therefore II_1 \cdot II_2 \cdot II_3 = (2R \sin A) (2R \sin B) (2R \sin C) \sec \frac{A}{2} \sec \frac{B}{2} \sec \frac{C}{2}$$

$$= 8R^3 \cdot \frac{\left(2 \sin \frac{A}{2} \cos \frac{A}{2}\right) \left(2 \sin \frac{B}{2} \cos \frac{B}{2}\right) \left(2 \sin \frac{C}{2} \cos \frac{C}{2}\right)}{\cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}}$$

$$= 64R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \quad \therefore r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\therefore II_1 \cdot II_2 \cdot II_3 = 16R^2 r \quad \text{Hence Proved}$$

$$(ii) II_1^2 + I_2 I_3^2 = II_2^2 + I_3 I_1^2 = II_3^2 + I_1 I_2^2$$

$$\therefore II_1^2 + I_2 I_3^2 = a^2 \sec^2 \frac{A}{2} + a^2 \operatorname{cosec}^2 \frac{A}{2} = \frac{a^2}{\sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}$$

$$\therefore a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2} \quad \therefore II_1^2 + I_2 I_3^2 = \frac{16R^2 \sin^2 \frac{A}{2} \cdot \cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2} \cdot \cos^2 \frac{A}{2}} = 16R^2$$

Similarly we can prove $II_2^2 + I_3 I_1^2 = II_3^2 + I_1 I_2^2 = 16R^2$

$$\text{Hence } II_1^2 + I_2 I_3^2 = II_2^2 + I_3 I_1^2 = II_3^2 + I_1 I_2^2$$

Self Practice Problem :

1. In a ΔABC , if $b = 2$ cm, $c = \sqrt{3}$ cm and $\angle A = \frac{\pi}{6}$, then find distance between its circumcentre and incentre.

Ans. $\sqrt{2 - \sqrt{3}}$ cm